AN EXACT SOLUTION OF AN AXIALLY SYMMETRIC PROBLEM OF IDEAL PLASTICITY

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We obtain an exact solution of a static axially symmetric problem of ideal plasticity with the von Mises condition of plasticity.

Let r, φ, z be cylindrical coordinates, let $\sigma_r \sigma_\varphi$, σ_z, τ_{rz} be the components of the stress tensor, let u and w be the vector velocity components, and let k be a constant. The stress tensor components, which are independent of φ, and the vector velocity components satisfy the equations

$$\begin{gathered}\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \quad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \\ (\sigma_r - \sigma_\varphi)^2 + (\sigma_\varphi - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6\tau_{rz}^2 = 6k^2 \\ \frac{\partial u}{\partial r} = \lambda (2\sigma_r - \sigma_\varphi - \sigma_z), \quad \frac{u}{r} = \lambda (2\sigma_\varphi - \sigma_z - \sigma_r) \\ \frac{\partial w}{\partial z} = \lambda (2\sigma_z - \sigma_r - \sigma_\varphi), \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 6\lambda \tau_{rz}\end{gathered} \tag{1}$$

Here λ is a positive function.

We seek a solution of the system (1) in the form

$$\begin{gathered}\sigma_r = az + \sigma_r^*, \quad \sigma_\varphi = az + \sigma_\varphi^*, \quad \sigma_z = az + \sigma_z^* \\ u = u^* \exp z, \quad w = w^* \exp z\end{gathered} \tag{2}$$

Here a is an arbitrary constant and the starred quantities depend only on r.

From Eqs. (1) it follows that

$$\tau_{rz} = -{}^1\!/_2 ar + cr^{-1} \tag{3}$$

where c is an arbitrary constant.

Substituting the relations (2) and (3) into Eqs. (1) and eliminating λ, we obtain a system of ordinary differential equations for the starred quantities.

We consider the case $a = c = 0$. From the condition of incompressibility

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

which follows from Eqs. (1), and the condition $\tau_{rz} = 0$, we find, assuming that w is bounded for r = 0,

$$u = AJ_0'(r) \exp z, \quad w = -AJ_0(r) \exp z \tag{4}$$

[If w is not bounded for r = 0, a MacDonald function must be added in Eqs. (4).]

In the expressions (4) A is an arbitrary constant,

$$J_0'(t) \equiv dJ_0(t) / dt$$

$J_0(t)$ is the zero-order Bessel function of an imaginary argument, satisfying the equation

$$t d^2 J_0(t) / dt^2 + dJ_0(t) / dt - tJ_0(t) = 0$$

and the conditions

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$$J_0(0) = 1, \quad J_0'(0) = 0$$

Let

$$-rJ_0(r) / J_0'(r) \equiv f(r)$$

Here $f(r)$ assumes a value from -2 to ∞ if r varies from 0 to ∞. We obtain [1], p. 304,

$$\sigma_r / k = -\int_r^R [f(r) + 2] r^{-1} F(r)\, dr, \quad F(r) \equiv [1 + f(r) + f^2(r)]^{-1/2}$$
$$\sigma_\varphi / k = \sigma_r / k + [f(r) + 2] F(r) \qquad (5)$$
$$\sigma_z / k = \sigma_r / k + [2f(r) + 1] F(r)$$

where R is an arbitrary constant.

For $R > 0$ and $A > 0$ this solution describes the plastic flow of a circular cylinder of length L ($-L \leq z \leq 0$, $0 \leq r \leq R$), stress-loaded at its flat ends according to the law (5), and stress-free on its lateral surface.

We remark that the system (1), with λ eliminated, admits a Lie algebra of operators [2, 3] with the basis

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial w}$$
$$X_4 = u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}, \quad X_5 = \frac{\partial}{\partial \sigma_r} + \frac{\partial}{\partial \sigma_\varphi} + \frac{\partial}{\partial \sigma_z}$$

This algebra is a Lie subalgebra of operators admitted by the system (1). The solution of the form (2), and also the solution given in [4], p. 96, are invariant solutions constructed on one-dimensional subgroups generated by the operators (5).

LITERATURE CITED

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